The Twenty Third Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities

Brenna, Poland, January 31 – February 3, 2024

List of Participants and talks

Roman Badora, *Searching for new versions of the Kranz separation theorem*

Mihály Bessenyei, *Existence theorems for invariance equations*

Zoltán Boros, *An alternative equation involving two generalized monomials*

Jacek Chmieliński, *Alternate additivity of the Birkhoff-James orthogonality*

Jacek Chudziak, *Risk diversification with the zero utility principle*

Attila Gilányi, *Determining types of functional equations with computer*

Richárd Grünwald, *Properties of the set of solutions of the global comparison problem of Gini means*

Eszter Gselmann, *A characterization of differential operators in the ring of complex polynomials*

Mehak Iqbal, *Quadratic functions as solutions of polynomial equations*

Justyna Jarczyk, *Characterization of complex-valued exponential functions via an iterative functional equation*

Witold Jarczyk, *Extension theorem for simultaneous q-difference equation and some its consequences*

Tibor Kiss, *On a non-symmetric version of the drop theorem*

Radosław Łukasik, *Definition and properties of a fuzzy Xor*

Rayene Menzer, *An alternative equation for polynomial functions on locally compact Abelian groups*

Gábor Marcell Molnár, *On approximate convexity*

Gergő Nagy, *Points of operator convexity of functions on operator algebras*

Andrzej Olbryś, *On approximate convexity*

Zsolt Páles, *Taylor-type theorems with respect to Chebyshev systems*

Paweł Pasteczka, *Multivariable generalizations of bivariate means via invariance*

Patryk Rela, *The Orlicz premium principle under uncertainty*

Maciej Sablik, *Generalized discount factors*

Justyna Sikorska, *On a characterization of the logarithmic mean*

László Székelyhidi, *On the Spectral Synthesis Theorem of Laurent Schwartz*

Patrícia Szokol, *Some results and open questions on quasi-arithmetic means*

Tomasz Szostok, *Inequalities for 2-convex functions involving signed measures*

Lan Nhi To, *Computer assisted investigation of Levi-Civita type functional equations*

Norbert Tóth, *The coincidence set of generalized monotone functions*

Péter Tóth, *On measurable solutions of an alternative functional equation*

Paweł Wójcik, *On an orthogonality equation in finite-dimesional normed spaces*

Sebastian Wójcik, *Comonotonic additivity of the zero utility principle under uncertainty*

Programme

Thursday

8:00 - 9:00	Breakfast
9:05 - 9:10	**Maciej Sablik** Opening
	First morning session — Chair: **Eszter Gselmann**
9:10 - 9:30	**Jacek Chudziak** Risk diversification with the zero utility principle
9:35 - 9:55	**Patryk Rela** The Orlicz premium principle under uncertainty
10:00 - 10:20	**Sebastian Wójcik** Comonotonic additivity of the zero utility principle under uncertainty
10:25 - 10:55	Coffee break
	Second morning session — Chair: **Jacek Chudziak**
10:55 - 11:15	**Mihály Bessenyei** Existence theorems for invariance equations
11:20 - 11:40	**Norbert Tóth** The coincidence set of generalized monotone functions
11:45 - 12:05	**Roman Badora** Searching for new versions of the Kranz separation theorem
13:00 - 14:00	Lunch
	First afternoon session — Chair: **Jacek Chmieliński**
15:00 - 15:20	**Eszter Gselmann** A characterization of differential operators in the ring of complex polynomials
15:25 - 15:45	**Mehak Iqbal** Quadratic functions as solutions of polynomial equations
15:50 - 16:10	**Gergő Nagy** Points of operator convexity of functions on operator algebras
16:15 - 16:30	Coffee break
	Second afternoon session — Chair: **László Székelyhidi**
16:30 - 16:50	**Jacek Chmieliński** Alternate additivity of the Birkhoff-James orthogonality
16:55 - 17:15	**Paweł Wójcik** On an orthogonality equation in finite-dimesional normed spaces
17:20 - 17:45	**Justyna Sikorska** On a characterization of the logarithmic mean
18:00 - 19:00	Dinner

Friday

8:00 - 9:00	Breakfast
	First morning session Chair: **Witold Jarczyk**
9:00 - 9:20	**László Székelyhidi** On the Spectral Synthesis Theorem of Laurent Schwartz
9:25 - 9:45	**Zoltán Boros** An alternative equation involving two generalized monomials
9:50 - 10:10	**Rayene Menzer** An alternative equation for polynomial functions on locally compact Abelian groups
10:15 - 10:45	Coffee break
	Second morning session Chair: **Attila Gilányi**
10:45 - 11:05	**Justyna Jarczyk** Characterization of complex-valued exponential functions via an iterative functional equation
11:10 - 11:30	**Tomasz Szostok** Inequalities for 2-convex functions involving signed measures
11:35 - 11:55	**Patrícia Szokol** Some results and open questions on quasi-arithmetic means
13:00 - 14:00	Lunch
	First afternoon session Chair: **Zoltán Boros**
15:00 - 15:20	**Witold Jarczyk** Extension theorem for simultaneous q-difference equation and some its consequences
15:25 - 15:45	**Attila Gilányi** Determining types of functional equations with computer
15:50 - 16:10	**Lan Nhi To** Computer assisted investigation of Levi-Civita type functional equations
16:15 - 16:30	Coffee break
	Second afternoon session Chair: **Justyna Jarczyk**
16:30 - 16:50	**Zsolt Páles** Taylor-type theorems with respect to Chebyshev systems
16:55 - 17:15	**Richárd Grünwald** Properties of the set of solutions of the global comparison problem of Gini means
17:20 - 17:40	**Paweł Pasteczka** Multivariable generalizations of bivariate means via invariance
17:40 - 18:10	**Problems and remarks**
19:00	Festive dinner

Saturday

8:00 - 9:00	Breakfast	
	First morning session	Chair: **Mihály Bessenyei**
9:00 - 9:20	**Andrzej Olbryś**	
	On approximate convexity	
9:25 - 9:45	**Gábor Marcell Molnár**	
	On approximate convexity	
9:50 - 10:10	**Tibor Kiss**	
	On a non-symmetric version of the drop theorem	
10:15 - 10:45	Coffee break	
	Second morning session	Chair: **Tomasz Szostok**
10:45 - 11:05	**Radosław Łukasik**	
	Definition and properties of a fuzzy Xor	
11:10 - 11:30	**Péter Tóth**	
	On measurable solutions of an alternative functional equation	
11:35 - 11:55	**Maciej Sablik**	
	Generalized discount factors	
12:00 - 12:10	**Zsolt Páles**	
	Closing	
12:30 - 13:30	Lunch	

Abstracts

Searching for new versions of the Kranz separation theorem

Roman Badora
University of Silesia

Suppose we have two real functionals defined on a commutative semigroup (or group) and one of them lies below the other. During the talk, we will analyze which of the systems of two inequalities describing the relationship between the values of these functionals on the sum of arguments and the sum of their values on these arguments guarantees the separation of the given functionals by an additive mapping.

Existence theorems for invariance equations

Mihály Bessenyei
University of Debrecen/Miskolc
(joint work with Evelin Pénzes)

The Kuratowski measure of noncompactness provides direct approach to the Sadovskiĭ fixed point theorem or to Hutchinson's fundamental result concerning fractals. It turns out that this measure is not distinguished: Requiring quite simple properties on a set-function, we can prove analogous of these results. The common idea behind is an abstract domain invariance property which can be justified with the Knaster–Tarski and the Kantorovitch Fixed Point Theorems.

An alternative equation involving two generalized monomials

Zoltán Boros
University of Debrecen

(joint work with Rayene Menzer)

In this presentation we consider generalized monomials or polynomials f, $g : \mathbb{R} \to \mathbb{R}$ that satisfies the additional equation $f(x)g(y) = 0$ for the pairs $(x, y) \in D$, where $D \subset \mathbb{R}^2$ is given by some algebraic condition. In the particular cases when f and g are generalized polynomials and there exist non-constant regular polynomials p and q that fulfill

$$D = \{\, (p(t), q(t)) \,|\, t \in \mathbb{R} \,\}$$

or f and g are generalized monomials and there exists a non-zero rational m fulfilling

$$D = \{\, (x, y) \in \mathbb{R}^2 \,|\, x^2 - my^2 = 1 \,\},$$

we prove that either f or g is identically equal to zero.

Our research is motivated by such results for $g = f$ in [1] and [2].

References

[1] Z. Boros, W. Fechner, *An alternative equation for polynomial functions*, Aequationes Math. **89/1** (2015), 17–22.

[2] Z. Boros, R. Menzer, *An alternative equation for generalized monomials*, Aequationes Math. **97** (2023), 113–120.

Alternate additivity of the Birkhoff-James orthogonality

Jacek Chmieliński
University of the National Education Commission, Krakow
(joint work with Paweł Wójcik)

The Birkhoff-James orthogonality $\perp_B$ is not additive (neither on the right nor on the left) unless certain additional geometrical properties (like smoothness, strict convexity or inner product structure) are imposed on the underlying space. We establish some weaker forms of the said additivity which are true without any additional assumptions. In particular, we will show that for a real normed space and arbitrary vectors x, y, z, we always have the alternative:

$$x \perp_B y \quad \text{and} \quad x \perp_B z \quad \Longrightarrow \quad x \perp_B (y+z) \quad \text{or} \quad x \perp_B (y-z).$$

For the left-additivity, the situation is more complex. If the underlying space is a two-dimensional real normed space, then we have for all x, y, z:

$$y \perp_B x \quad \text{and} \quad z \perp_B x \quad \Longrightarrow \quad (y+z) \perp_B x \quad \text{or} \quad (y-z) \perp_B x.$$

If the dimension of the considered space is greater than two, the latter condition characterizes inner product spaces among all smooth or strictly convex real normed spaces.

Risk diversification with the zero utility principle

Jacek Chudziak
University of Rzeszów
(joint work with Paweł Pasteczka and Patryk Rela)

Let $\mathcal{X}_+$ be a family of risks, that is non-negative essentially bounded random variables on a given probability space. A zero utility premium for $X \in \mathcal{X}_+$, introduced by Bühlmann [1], is defined through the equation

$$E[u(H_u(X) - X)] = 0, \tag{1}$$

where $u : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function such that $u(0) = 0$. In [2] characterizations of various important properties of the zero utility principle defined by (1) was established. In particular, it was proved there that the principle is convex if and only if u is concave. This result has been extended in [3], where it was shown that H_u is quasi-convex, that is

$$H_u\left(\frac{X+Y}{2}\right) \le \max\{H_u(X), H_u(Y)\} \quad \text{for} \quad X, Y \in \mathcal{X}_+,$$

if and only if it is convex. Thus, a quasi-convexity of H_u is equivalent to concavity of u. It is known that the quasi-convexity has the following natural interpretation: a premium for a portfolio composed of two risks using the arithmetic mean does not exceed a maximum of the premiums for the individual risks. Motivated by the above results, for a given function u we are interested in functions $f : [0, \infty)^2 \to [0, \infty)$ allowing the risk diversification, that is for which the following inequality is satisfied

$$H_u(f(X, Y)) \le \max\{H_u(X), H_u(Y)\} \quad \text{for} \quad X, Y \in \mathcal{X}_+. \tag{2}$$

In our investigations we apply some results concerning properties of the quasideviation means, proved by Páles [4].

REFERENCES

[1] Bühlmann, H. *Mathematical Models in Risk Theory* Springer-Verlag, Berlin, 1970.
[2] Chudziak, J. On applications of inequalities for quasideviation means in actuarial mathematics. *Math. Inequal. Appl.* **2018**, *21*, 3, 601–610.
[3] Chudziak, J. On quasi-convexity of the zero utility principle *J. Nonlinear Convex Anal.* **2018**, *19*, 5, 749–758.
[4] Páles, Zs. General inequalities for quasideviation means. *Aequationes Math.* **1988**, *36*, 1, 32–56.

Determining types of functional equations with computer

Attila Gilányi
University of Debrecen
(joint work with Lan Nhi To)

Nowadays, computer assisted investigations play an increasingly important role connected to studies of functional equations, inequalities and related topics (cf., e.g., the papers [1], [2], [3], [4] and the references therein). In this talk, we present a package of computer programs developed in the computer algebra system MAPLE, which is able to decide about certain functional equations to which class of functional equations they belong and (if applicable) it can determine their type as well.

References

[1] Gergő Gyula Borus, Attila Gilányi, *Computer assisted solution of systems of two variable linear functional equations*, Aequationes Math. **94** (2020), 723–736.

[2] Attila Gilányi, Lan Nhi To, *Computer assisted investigation of alienness of linear functional equations*, Aequationes Math. **97** (2023), 1185–1199.

[3] Chisom Prince Okeke, Maciej Sablik, *Functional equation characterizing polynomial functions and an algorithm*, Results Math. **77** (2022), Paper No. 125, 17.

[4] Chisom Prince Okeke, Wisdom I. Ogala, Timothy Nadhomi, *On symbolic computation of C. P. Okeke functional equations using Python programming language*, Aequationes Math., accepted for publication.

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Properties of the set of solutions of the global comparison problem of Gini means

Richárd Grünwald
University of Nyíregyháza and Doctoral School, University of Debrecen
(joint work with Zsolt Páles)

Let us recall the definition of the *n-variable Gini mean corresponding to the pair parameters* $(p,q) \in \mathbb{R}^2$:

$$G_{p,q}^{[n]}(x_1,\dots,x_n) := \begin{cases} \left(\dfrac{x_1^p+\cdots+x_n^p}{x_1^q+\cdots+x_n^q}\right)^{\frac{1}{p-q}} & \text{if } p \neq q \\ \exp\left(\dfrac{x_1^p\ln(x_1)+\cdots+x_n^p\ln(x_n)}{x_1^p+\cdots+x_n^p}\right) & \text{if } p = q, \end{cases} \qquad (x_1,\dots,x_n \in \mathbb{R}_+).$$

Let us consider the global comparison problem of Gini means with fixed number of variables in a subinterval I of $\mathbb{R}_+$, i.e., the following inequality

$$G_{r,s}^{[n]}(x_1,\dots,x_n) \le G_{p,q}^{[n]}(x_1,\dots,x_n), \tag{1}$$

where $n \in \mathbb{N}, n \ge 2$ is fixed, $(p,q),(r,s) \in \mathbb{R}^2$ and $x_1,\dots,x_n \in I$.

Given a nonempty subinterval I of $\mathbb{R}_+$ and $n \in \mathbb{N}$, we introduce the sets

$$\Gamma_n(I) := \{((r,s),(p,q)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid (1) \text{ holds for all } x_1,\dots,x_n \in I\},$$
$$\Gamma_\infty(I) := \bigcap_{n=1}^{\infty} \Gamma_n(I).$$

In the talk, we will investigate the properties of these sets and their relationship to each other.

References

[1] Zoltán Daróczy and László Losonczi, *Über den Vergleich von Mittelwerten*, Publ. Math. Debrecen **17** (1970), 289–297.
[2] Zsolt Páles, *Inequalities for sums of powers*, J. Math. Anal. Appl. **131** (1988), 265–270.
[3] Zsolt Páles, *On comparison of homogeneous means*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **32** (1989), 261–266.
[4] Zsolt Páles, *Comparison of two variable homogeneous means*, Int. Series of Num. Math. (1992), 59–70.

A characterization of differential operators in the ring of complex polynomials

Eszter Gselmann
University of Debrecen

(joint work with Włodzimierz Fechner)

This talk aims to provide a full characterization of all operators $T\colon \mathscr{P}(\mathbb{C}) \to \mathscr{P}(\mathbb{C})$ acting on the space of all complex polynomials that satisfy the Leibniz rule

$$T(f \cdot g) = T(f) \cdot g + f \cdot T(g)$$

for all $f, g \in \mathscr{P}(\mathbb{C})$. We do not assume the linearity of T. As we will see, contrary to the well-known theorems for function spaces there are many other solutions here, not only differential operators. From our main result, we also derive two corollaries, showing that in some special cases operators that satisfy the Leibniz rule have some particular form.

Quadratic functions as solutions of polynomial equations

Mehak Iqbal
University of Debrecen
(joint work with Eszter Gselmann)

Polynomial equations play a significant role in algebra and the theory of functional equations. If the unknown functions in the equation are additive, relatively many results are known. In some specific cases, according to classical results, the unknown additive functions are homomorphisms, derivations, or linear combinations of these. Now the question arises whether the solutions can be described even if the unknown functions are not assumed to be additive but to be generalized monomials. As a starting point, we will deal with generalized monomials of degree two, that is, with quadratic functions. Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{F} \subset \mathbb{K}$ be a subfield of $\mathbb{K}$. Our main objective is to determine all those quadratic functions $q \colon \mathbb{F} \to \mathbb{K}$ that satisfy a Levi-Civita equation on the multiplicative structure, i.e., that can be written as

$$q(xy) = \sum_{i=1}^{k} g_i(x)h_i(y) \qquad \left(x, y \in \mathbb{F}^{\times}\right)$$

with some positive integer k and with some appropriate functions g_i, h_i, $i = 1, \ldots, k$. For this, those quadratic functions q that satisfy the equations

$$q(xy) = q(x)q(y) \qquad \left(x, y \in \mathbb{F}^{\times}\right)$$

and

$$q(xy) = x^2 q(y) + q(x)y^2 \qquad \left(x, y \in \mathbb{F}^{\times}\right),$$

respectively, must first be determined.

Characterization of complex-valued exponential functions via an iterative functional equation

Justyna Jarczyk
University of Zielona Góra

This is a report on the research made jointly with Witold Jarczyk.

Fix a positive integer $n \geq 2$ and a number $a \in (0, +\infty]$. Let $f_1, \ldots, f_n$ be selfmappings of the interval $(0, a)$ summing up to the identity function:

$$\sum_{j=1}^{n} f_j(x) = x, \quad x \in (0, a).$$

Given an $(n-1)$-th root $\omega \in \mathbb{C}$ of unity and a complex number c, and defining $\psi_{\omega,c} : (0, a) \to \mathbb{C}$ by $\psi_{\omega,c}(x) = \omega \exp(cx)$, we see that

$$\begin{aligned} \psi_{\omega,c}(x) &= \omega \exp\left(c \sum_{j=1}^{n} f_j(x)\right) = \omega \prod_{j=1}^{n} \exp\left(c f_j(x)\right) \\ &= \frac{\omega}{\omega^n} \prod_{j=1}^{n} \psi_{\omega,c}\left(f_j(x)\right) = \prod_{j=1}^{n} \psi_{\omega,c}\left(f_j(x)\right) \end{aligned}$$

for all $x \in (0, a)$. Therefore $\psi_{\omega,c}$ satisfies the functional equation

$$\psi(x) = \prod_{j=1}^{n} \psi\left(f_j(x)\right).$$

During the talk I am going to prove that under some assumptions also the converse is true.

Extension theorem for simultaneous q-difference equations and some its consequences

Witold Jarczyk
University of Zielona Góra

The presented results have been obtained jointly with Paweł Pasteczka.

Given a set $T \subset (0, +\infty)$, intervals $I \subset (0, +\infty)$ and $J \subset \mathbb{R}$, as well as functions $g_t : I \times J \to J$ with t's running through the set

$$T^* := T \cup \left\{t^{-1} : t \in T\right\} \cup \{1\}$$

we study the simultaneous q-difference equations

$$\varphi(tx) = g_t\left(x, \varphi(x)\right), \qquad t \in T^*,$$

postulated for $x \in I \cap t^{-1}I$; here the unknown function φ is assumed to map I into J. We present an extension theorem stating that if φ is continuous [analytic] on a nontrivial subinterval of I, then φ is continuous [analytic] provided $g_t, t \in T^*$, are continuous [analytic]. The crucial assumption of the extension theorem is formulated with the help of the so-called limit ratio R_T which is a uniquely determined number from $[1, +\infty]$, characterising some density property of the set T^*. As an application of the extension theorem we find the form of all continuous on a subinterval of I solutions $\varphi : I \to \mathbb{R}$ of the simultaneous equations

$$\varphi(tx) = \varphi(x) + c(t)x^p, \qquad t \in T,$$

where $c : T \to \mathbb{R}$ is an arbitrary function, p is a given real number and $\sup I > R_T \inf I$.

On a non-symmetric version of the drop theorem

Tibor Kiss
University of Debrecen

As is widely known, the convex hull of the union of a convex subset and a point of a linear space equals to the union of the segments starting at the given point and ending in the set in question. This result is called the *drop theorem*. In the talk we will restrict ourselves to the real number line and deal with a variant of this result.

For a fixed parameter $t \in [0, 1]$, we say that a subset $D \subseteq \mathbb{R}$ is *non-symmetrically t-convex* if $tx + (1 - t)y \in D$ whenever $x, y \in D$ with $y \leq x$. To avoid the trivial cases, we also assume that $t \notin \{0, \frac{1}{2}, 1\}$.

In the talk we give a sufficient condition under which the non-symmetric t-convex hull of a non-symmetric t-convex segment and a point outside it can be represented in the way detailed above.

References

[1] M. Lewicki and A. Olbryś, *On non-symmetric t-convex functions*, Math. Inequal. Appl.17(2014), no.1, 95–100.
[2] K. Nikodem and Zs. Páles, *Note on t-quasiaffine functions*, Ann. Univ. Sci. Budapest. Sect. Comput.29(2008), 127–139.

Definition and properties of a fuzzy Xor

Radosław Łukasik
University of Silesia

In this talk we show that the fuzzy Xor defined in [1] cannot have some properties presented in that paper. We also provide new constructions of fuzzy Xor based on the composition of other fuzzy connectives.

REFERENCE

[1] B.C. Bedregal, G.P. Dimuro, R.H.S. Reiser, Revisiting Xor-implications: Classes of fuzzy (co)implications based on f-Xor (f-XNor) connectives, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 21 (2013), 899–925.

An alternative equation for polynomial functions on locally compact Abelian groups

Rayene Menzer
University of Debrecen
(joint work with Zoltán Boros)

In our presentation we establish the following result:

Theorem. *Let G be a locally compact Abelian group which is generated by any neighborhood of zero. Let μ denote the Haar measure on G, and let us assume that μ is σ-finite. Let $f : G \to \mathbb{C}$ be a generalized polynomial fulfilling*

$$f(x)f(y) = 0 \tag{1}$$

for all $(x, y) \in D$, where $D \subseteq G^2$ is a $\mu \times \mu$ measurable subset with positive measure. Then $f(x) = 0$ for every $x \in G$.

This research is motivated by the particular case in [3] when $G = \mathbb{R}^k$ for some natural number k and f is additive, as well as by similar investigations in [1] for real generalized polynomials with a particular algebraic constraint (namely, when D is the unit circle). The main tool is provided by Székelyhidi's results [4] on the zeros of generalized polynomials in an abstract setting. A particular case of our main theorem is in print [2].

References

[1] Z. Boros, W. Fechner, *An alternative equation for polynomial functions*, Aequationes Math. **89/1** (2015), 17–22.
[2] Z. Boros, R. Menzer, *An alternative equation for generalized monomials involving measure*, Publ. Math. Debrecen (accepted; to appear in April, 2024).
[3] Z. Kominek, L. Reich and J. Schwaiger, *On additive functions fulfilling some additional condition*, Sitzungsber. Abt. II **207** (1998), 35–42.
[4] L. Székelyhidi, *Regularity properties of polynomials on groups*, Acta Math. Hung. **45** (1985), 15–19.

On approximate convexity

Gábor Marcell Molnár
University of Nyíregyháza and University of Debrecen
(joint work with Zsolt Páles)

Let X be a real linear space, $D \subseteq X$ nonempty, convex and $D_\Delta := \{x - y : x, y \in D\}$. Let $\varphi : \frac{1}{2}D_\Delta \to \mathbb{R}$ be a given function, called an *error function*. We say that a function $f\colon D \to \mathbb{R}$ is *φ-Jensen convex on D* (or *φ-midconvex on D*) if

$$f\Big(\frac{x+y}{2}\Big) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y) + \varphi\Big(\frac{x-y}{2}\Big) \qquad (x, y \in D).$$

The basic problem related to a φ-Jensen convex function $f\colon D \to \mathbb{R}$ is to deduce further approximate convexity properties.

In the talk I am aiming to present an approach to find approximate convexity properties of a φ-Jensen convex function.

Reference

[1] Judit Makó and Zsolt Páles, *On φ-convexity*, Publ. Math. Debrecen, **80**(1-2):107–126, 2012.

Points of operator convexity of functions on operator algebras

Gergő Nagy
University of Debrecen

In 2010, Silvestrov, Osaka and Tomiyama verified that a C^*-algebra $\mathcal{A}$ is commutative exactly when there exists a continuous function $f\colon [0,\infty[\to \mathbb{R}$ which is not convex on the set of all positive semidefinite 2×2 matrices but convex on the collection of all positive elements in $\mathcal{A}$, i.e. $\mathcal{A}$-convex. As a local version of this theorem, Virosztek showed that in certain cases, the "points of operator convexity" of convex, but not $\mathcal{A}$-convex functions are precisely the central elements of the algebra. In the talk, after a brief overview of some related former results, we present the following generalization of this statement. If $D\subset\mathbb{R}$ is an open interval and $f\in\mathcal{C}^2(D)$ is a convex function satisfying a certain technical condition, and $a\in\mathcal{A}$ is a self-adjoint element, then a is central if and only if it is a point of operator convexity of f.

On approximate convexity

Andrzej Olbryś
University of Silesia

Let D be a convex subset of a real linear space X. In this talk we examine the properties of functions $f : D \to \mathbb{R}$ satisfying the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \phi(t(x-y)) - t\phi(x-y),$$

for all $x, y \in D,\ t \in [0,1]$, where $\phi : X \to \mathbb{R}$ is a given function.

Taylor-type theorems with respect to Chebyshev systems

Zsolt Páles
University of Debrecen

The aim of the talk is to present an exact error formula for the Taylor-type interpolation of smooth functions in terms of Chebyshev systems. The main tool to achieve this goal is the following easy-to-prove result:

Theorem. Let $I \subseteq \mathbb{R}$ be a nondegenerate interval and let $A : I \to \mathbb{R}^{n\times n}$ be a continuous matrix-valued function. Assume that $Y : I \to \mathbb{R}^{n\times n}$ is a matrix-valued solution of the linear differential equation

$$Y'(x) = A(x)Y(x) \qquad (x \in I)$$

such that $Y(x)$ is nonsingular for all $x \in I$. Then, for all continuosly differentiable functions $f : I \to \mathbb{R}^n$ and for all $a, x \in I$, the equality

$$f(x) = Y(x)\bigg(Y^{-1}(a)f(a) + \int_a^x Y^{-1}(t)\big(f'(t) - A(t)f(t)\big)dt\bigg)$$

holds.

Multivariable generalizations of bivariate means via invariance

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For a given p-variable mean $M\colon I^p \to I$ (I is a subinterval of $\mathbb{R}$), following Horwitz and Lawson–Lim, we can define (under certain assumption) its $(p+1)$-variable β-invariant extension as the unique solution $K\colon I^{p+1} \to I$ of the functional equation

$$K\big(M(x_2,\dots,x_{p+1}), M(x_1,x_3,\dots,x_{p+1}),\dots,M(x_1,\dots,x_p)\big) \\ = K(x_1,\dots,x_{p+1}), \text{ for all } x_1,\dots,x_{p+1}\in I$$

in the family of means.

Applying this procedure iteratively we can obtain a mean which is defined for vectors of arbitrary lengths starting from the bivariate one. The aim of this talk is to study the properties of such extensions.

References

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The Orlicz premium principle under uncertainty

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(joint work with Jacek Chudziak)

Under the expected utility model the Orlicz premium principle for a risk X, represented by a non-negative essentially bounded random variable on a given probability space, is defined implicitly, as a unique solution $H_{\alpha,\Phi}(X)$ of the equation

$$E\left[\Phi\left(\frac{X}{H_{\alpha,\Phi}(X)}\right)\right] = 1-\alpha, \tag{1}$$

where $\alpha \in [0,1)$ is a given parameter and $\Phi : [0,\infty) \to [0,\infty)$ is a normalized Young function, that is a strictly increasing, convex function $\Phi : [0,\infty) \to [0,\infty)$ satisfying $\Phi(0) = 0$, $\Phi(1) = 1$ and $\lim_{x\to\infty} \Phi(x) = \infty$. The Orlicz premium in this setting has been introduced by [2]. Several details concerning properties of the premium defined by (1) can by found in [1].

In order to define the Orlicz premium principle under uncertainty, assume that $(\Omega, \mathcal{F})$ is a measurable space and $\mu : \mathcal{F} \to [0,1]$ is a capacity, that is a monotone set function satisfying $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$. Let $\mathcal{X}_+$ be a family of all $\mathcal{F}$-measurable functions $X : \Omega \to [0,\infty)$ such that $\mu(\{X > t\}) = 0$ for some $t \in \mathbb{R}$. The premium for $X \in \mathcal{X}_+$ is defined through the equation

$$E_\mu\left[\Phi\left(\frac{X}{H_{\mu,\alpha,\Phi}(X)}\right)\right] = 1-\alpha, \tag{2}$$

where $\alpha \in [0,1)$, $\Phi : [0,\infty) \to [0,\infty)$ is a normalized Young function and

$$E_\mu[X] = \int_0^\infty \mu(\{X > x\})dx \quad \text{for} \quad X \in \mathcal{X}_+,$$

is the Choquet integral with respect to the capacity μ.

The aim of this talk is to prove the existence and uniqueness of the Orlicz premium defined by (2) and to characterize its several important properties.

References

[1] Bellini, F., Gianin, E. R. On Haezendonck risk measures. *Journal of Banking and Finance* **2007**, *32*, 986–994.

[2] Haezendonck, J., Goovaerts, M. A new premium calculation principle based on Orlicz norms, *Insurance: Mathematics and Economics* **1982**, *1*, 41–53.

Generalized discount factors

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We consider the so called generalized discount factors, i.e. nonincreasing functions $\phi : \mathbb{N} \to [0,1]$, satisfying $\phi(0) = 1$ and $\phi(n+k) \geq \phi(n)\phi(k)$, for $n, k \in \mathbb{N}$. Typical example is a generalized *hyperbolic discount factor* given by $\phi(i) = (1+hi)^{-\frac{r}{h}}$ with $h > 0$, $r > 0$ and $\frac{r}{h} \leq 1$. The discount factors appear in the problems of long run stochastic control (see eg. Łukasz Stettner [1]).

REFERENCE

[1] Łukasz Stettner, *Long run stochastic control problems with general discounting*, Mathematics of Operational Research, submitted.

On a characterization of the logarithmic mean

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(joint work with Timothy Nadhomi and Maciej Sablik)

Let $f\colon I \to \mathbb{R}$ and φ be an increasing function defined on the range of f. The function f is said to be φ-convex whenever $\varphi \circ f$ is convex, that is, for all $x, y \in I$, $t \in [0,1]$,

$$\varphi(f(tx + (1-t)y)) \leq t\varphi(f(x)) + (1-t)\varphi(f(y)),$$

and if φ is one-to-one,

$$f(tx + (1-t)y) \leq \varphi^{-1}(t\varphi(f(x)) + (1-t)\varphi(f(y))).$$

Starting from the celebrated Hermite-Hadamard inequality for φ-convex functions, we give some characterization of the logarithmic mean.

On the Spectral Synthesis Theorem of Laurent Schwartz

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In this talk we present a short proof for L. Schwartz's fundamental spectral synthesis theorem on the reals. The proof is based on our localization method and on the spectral analysis result proved by J. P. Kahane using the Carleman transform.

REFERENCES

[1] Laurent Schwartz, Théorie générale des fonctions moyenne-périodiques, *Ann. of Math. (2)* **48** (1947), 857–929.
[2] Jean-Pierre Kahane, *Lectures on mean periodic functions*, Tata Institute of Fundamental Research, Bombay 1959.

Some results and open questions on quasi-arithmetic means

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(joint work with Pál Burai and Gergely Kiss)

In this presentation, we focus the characterization theorem of János Aczél on quasi-arithmetic means. In his proof continuity is used essentially but a little bit furtively. We show that every bisymmetric, symmetric, reflexive, strictly monotonic binary map on a proper interval is continuous, in particular it is a quasi-arithmetic mean. Furthermore, we present some remarkable consequences of the previous result. We demonstrate that this result can be refined in the way that the symmetry condition can be weakened by assuming symmetry only for a pair of distinct points of an interval. Finally, concerning the obtained results we present some open questions.

Inequalities for 2-convex functions involving signed measures

Tomasz Szostok
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(joint work with Constantin P. Niculescu)

We present some remarks concerning problems posed in [1].

Reference

[1] D. Ş Marinescu, C. P. Niculescu, Old and new about 3−convex functions, https://arxiv.org/abs/2305.04353v1

Computer assisted investigation of Levi-Civita type functional equations

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(joint work with Attila Gilányi)

We consider Levi-Civita type functional equations

$$f(x+y)=\sum_{i=1}^{n} g_i(x)h_i(y), \tag{1}$$

where n is a positive integer, G is an Abelian group and $f, g_i, h_i : G \to \mathbb{C}$ $(i = 1, 2, \dots, n)$ are unknown functions.

Based on results by László Székelyhidi ([1]), we developed a computer program (written in the computer algebra system Maple) for determining the solution of functional equations of type (1).

In this talk, we present the Maple function with some demo examples of well-known Levi-Civita type functional equations and also in some cases when the right hand side of the input functional equation contains many terms.

Reference

[1] László Székelyhidi, *On the Levi-Civita Functional Equation*, Berichte der Mathematisch-Statistischen Sektion in der Forschungsgesellschaft Joanneum, 301. Forschungszentrum Graz, Mathematisch-Statistische Sektion, Graz (1988).

The coincidence set of generalized monotone functions

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(joint work with Mihály Bessenyei)

In a recent paper, Fu and Solow prove that the set of zeros of a convex function is either an interval or a finite set of at most two elements. Motivated by their result, we investigate the coincidence set of generalized lines and generalized convex functions, when the underlying notion is induced by an n-parameter Beckenbach family. It turns out that the situation in the extended context is quite similar to that of Fu and Solow: The coincidence set is either an interval or a finite set of at most n elements. Moreover, we show that the coincidence set can have k elements if $k \in [1, n] \cap \mathbb{N}$ and the family is an extended and complete Chebyshev-system.

REFERENCE

[1] F. Fu and D. Solow, *On the Roots of Convex Functions*, Journal of Convex Analysis **30** (2023), No. 1, 143–157.

On measurable solutions of an alternative functional equation

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Let I_1, I_2 be nonempty open intervals of the real line, and let $J := \frac{1}{2}(I_1 + I_2)$. The solutions of the functional equation

$$\varphi\left(\frac{x+y}{2}\right)(\psi_1(x) - \psi_2(y)) = 0 \qquad \text{(for all } x \in I_1 \text{ and } y \in I_2) \tag{1}$$

where the functions $\psi_1 : I_1 \longrightarrow \mathbb{R}$, $\psi_2 : I_2 \longrightarrow \mathbb{R}$ and $\varphi : J \longrightarrow \mathbb{R}$ are unknown, were described by T. Kiss [1]. It has been established that if $\varphi^{-1}(0)$ is closed then the nontrivial solutions of (1) are constant on some open subintervals of their domain.

During the Problems and Remarks session of the 59th International Symposium on Functional Equations Kiss proposed the following question (see [2]). Does the mentioned characterization of the solutions of (1) remain valid when the Darboux property is assumed for φ, instead of the closedness of $\varphi^{-1}(0)$? This is motivated by the fact that in certain applications (such as the invariance problem of generalized weighted quasi-arithmetic means) the functions appearing in (1) are derivatives, for which the set of zeros might not be closed.

In our talk we will present that unfortunately (1) has such nontrivial solutions $(\psi_1, \psi_2, \varphi)$ which are Darboux, yet neither function is constant on any open subinterval. On the other hand, we will show that if φ is measurable then an analogous version of the known characterization theorem for the solutions holds. Hence, if φ is supposed to be the derivative of a differentiable function, then (1) has exactly the same solutions as described in [1, Theorem 6.], which was desired for the applications.

References

[1] T. Kiss, *A Pexider equation containing the arithmetic mean*, Aequat. Math. (2023). https://doi.org/10.1007/s00010-023-00966-x

[2] Report of Meeting. *The 59th International Symposium on Functional Equations, Hotel Aurum, Hajdúszoboszló (Hungary), June 18–25, 2023.*, Aequat. Math. **97** (2023), 1259–1290.

On an orthogonality equation in finite-dimesional normed spaces

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(joint work with Karol Gryszka)

Let X, Y be real normed spaces and let ρ'_+, ρ'_- be norm derivatives. In this talk we consider a system of functional equations

$$\forall_{x,y\in X} \quad \begin{cases} \rho'_+\big(f(x), f(y)\big) = g(x)\rho'_+(x,y), \\ \rho'_-\big(f(x), f(y)\big) = g(x)\rho'_-(x,y), \end{cases}$$

with unknown functions $f\colon X \to Y$, $g\colon X \to \mathbb{R}$. As a consequence, we present partial answer to open problem posed in [2].

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Comonotonic additivity of the zero utility principle under uncertainty

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(joint work with Jacek Chudziak)

In a process of insurance contracts pricing, the insurance company assigns to any risk a non-negative real number, being a premium for the risk. There are various methods of insurance contracts pricing. In this talk we deal with a method, called the zero utility principle, introduced by H. Bühlmann (1970). This method presents the problem from the point of view of an insurance company, assuming that the premium for a given risk is determined in such a way that the company is indifferent between entering into contract and rejecting it.

We study the zero utility principle in the cumulative prospect theory (Tversky, Kahneman (1992)) under uncertainty. In this setting, the risks are represented by measurable functions defined on a given measurable space $(S, \mathcal{F})$. A premium for a risk X is defined as a unique real number $H_{(u,\mu,\nu)}(X)$ satisfying equation

$$E_{\mu\nu}[u(H_{(u,\mu,\nu)}(X) - X)] = 0,$$

where $u : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function such that $u(0) = 0$ and $E_{\mu\nu}$ is the Choquet integral with respect to a pair of capacities (μ, ν). Our main aim is to characterize comonotonic additivity of the principle. Recall that risks X and Y are comonotonic provided

$$(X(s_1) - X(s_2))(Y(s_1) - Y(s_2)) \geq 0 \quad \text{for} \quad s_1, s_2 \in S.$$

The premium is called additive for comonotonic risks if

$$H_{(u,\mu,\nu)}(X + Y) = H_{(u,\mu,\nu)}(X) + H_{(u,\mu,\nu)}(Y)$$

for any pair of comonotonic risks X and Y.

References

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[2] Tversky, A.; Kahneman, D. Advances in prospect theory: Cumulative representation of uncertainty. *J. Risk Uncertain.* **1992**, *5*, 297–323.

[3] Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R., Vyncke, D. The concept of comonotonicity in actuarial science and finance: theory *Insurance: Mathematics and Economics* **2002**, *31*, 1, 3–33.